Published by Institute of Physics Publishing for SISSA/ISAS
Received: *July 31, 2002*
Accepted: *August 16, 2002*

# Intersecting 6-branes from new 7-manifolds with $G_2$ holonomy

**Klaus Behrndt**
*Max-Plank-Institut für Gravitationsphysik, Albert Einstein Institut*
*Am Mühlenberg 1, 14476 Golm, Germany*
*E-mail:* behrndt@aei.mpg.de

**Gianguido Dall'Agata and Dieter Lüst**
*Humboldt Universität zu Berlin, Institut für Physik,*
*Invalidenstrasse 110, 10115 Berlin, Germany*
*E-mail:* dallagat@physik.hu-berlin.de, luest@physik.hu-berlin.de

**Swapna Mahapatra**
*Physics Department, Utkal University*
*Bhubaneswar 751 004, India*
*E-mail:* swapna@iopb.res.in



Abstract: We discuss a new family of metrics of 7-manifolds with $G_2$ holonomy, which are $\mathbb{R}^3$ bundles over a quaternionic space. The metrics depend on five parameters and have two abelian isometries. Certain singularities of the $G_2$ manifolds are related to fixed points of these isometries; there are two combinations of Killing vectors that possess co-dimension four fixed points which yield upon compactification only intersecting D6-branes if one also identifies two parameters. Two of the remaining parameters are quantized and we argue that they are related to the number of D6-branes, which appear in three stacks. We perform explicitly the reduction to the type IIA model.

Keywords: M-Theory, D-branes, Superstring Vacua, Differential and Algebraic Geometry.

## Contents





## 1. Introduction

It is well known that the compactification of M-theory (11-dimensional supergravity) on seven-manifolds $M_7$ of $G_2$ holonomy leads to an effective theory in four dimensions with $\mathcal{N} = 1$ supersymmetry. If $M_7$ is smooth, the harmonic Kaluza-Klein decomposition of the 11-dimensional massless degrees of freedom leads in four dimensions to $\mathcal{N} = 1$ supergravity coupled to abelian vector multiplets plus chiral multiplets, which correspond to the moduli of $M_7$ [1, 2]. On the other hand, if $M_7$ exhibits some singularities at certain points in the moduli space, massless non-abelian gauge bosons possibly together with massless chiral matter fields may emerge. The local neighborhood of these types of singularities can be best described by replacing the compact space $M_7$ by a non-compact $G_2$ manifold $X_7$ and we are essentially dealing with the geometric description of the effective low-energy gauge theory in four-dimensions (geometric engineering of gauge theories). In the following we are interested in M-theory on a non-compact background $X_7$ for which a number of examples have been discussed recently e.g. in [3]–[19].

If $X_7$ has a suitable U(1) isometry, one obtains a type IIA superstring interpretation upon dimensional reduction to ten dimensions. This circle is usually non-trivially fibred over a six-dimensional base $B_6$ which serves as the geometric background of the corresponding IIA superstring theory. In order to obtain non-abelian gauge groups with possibly chiral matter additional D6-branes have to wrap supersymmetric 3-cycles of $B_6$.

As a consequence, the gauge bosons correspond to open strings on the D6-brane world volumes, and chiral fermions arise from open strings stretching between different intersecting D6-branes. In this way, intersecting brane world models with intersecting D6-branes, being wrapped on homology 3-cycles of 6-dimensional tori, orbifolds or Calabi-Yau three-folds, can be constructed, which are more or less closely related to the standard model [20]–[28] (see e.g. [28] for a more complete list of references on intersecting brane world models). In M-theory language non-abelian gauge bosons arise, if $X_7$ has an A-D-E singularity of codimension four. The non-abelian gauge bosons correspond to massless M2-branes wrapped around collapsing 2-cycles and product gauge groups are provided by intersecting singularities. Massless fermions are supported by isolated (conical) singularities of codimension 7 of $X_7$ and this situation can be realized by two or more A-D-E singularities colliding into each other. In the IIA brane picture this is described by the intersection of D6-branes. One can also consider orientifold O6-planes (O6-planes correspond to the Atiyah-Hitchin manifold) intersected by D6-planes. E.g. an O6-plane intersected by $n$ D6-branes plus their mirror branes can lead to a SU($n$) gauge theory with chiral matter in the antisymmetric represenation of SU($n$). In M-theory this corresponds to unfold a $D_n$ singularity into a $A_{n-1}$ singularity.

Of course the IIA description depends very much on the choice of the U(1) action. In order to obtain a configuration that contains only D6-branes, one has to ensure that the 7-manifold has only co-dimension 4 fixed points and no co-dimension 2 and 6 fixed point sets. In this case, the 6-branes could be embedded in a topologically flat space and following the arguments given in [4, 9, 14] the topology of the 7-manifold should be completely encoded in the fixed point set of the U(1) action. In this case we can expect to describe a known 4-dimensional field theory living on the common intersection.

So far not many explicit metrics are known. Basically they group together into two classes [29, 30]: one is topologically a $\mathbb{R}^4$ bundle over $\mathbb{S}^3$ and the other a $\mathbb{R}^3$ bundle over a quaternionic base space. Many generalizations, with more parameters or functions, have been discussed in the past years. The first class e.g., can be generalized to $\mathbb{R}^4/\mathbb{Z}_N$ bundle over $\mathbb{S}^3$. In the second class one can consider further quaternionic spaces, different from e.g. the 4-sphere $\mathbb{S}^4$ and the complex projective space $\mathbb{CP}^2 = \mathrm{SU}(3)/\,\mathrm{U}(2)$, which are the only compact homogeneous quaternionic 4-dimensional spaces [31]. Apart from their non-compact analogs, there are also non-homogeneous quaternionic spaces as discussed in [15, 19, 16, 17]. For a closely related discussion of quaternionic spaces appearing in hyper Kaehler cones see [32, 33, 34].

In this paper we want to discuss a $G_2$ metric based on a quaternionic space with only two isometries. This 4-dimensional Einstein manifold can be obtained by a Wick rotation of a solution found by Demianski and Plebanski [35, 36] and is given by four roots of a fourth order polynomial. After some general comments about manifolds with $G_2$ and Spin(7) holonomy in the next section, we will discuss the quaternionic space and its symmetries in section 3. In section 4 we will discuss in detail the fixed point set of the two Killing vectors. Following the standard lore [7, 9, 14], we identify 6-branes as co-dimension four fixed points and avoid co-dimension two and six fixed point sets. Finally, in section 5 we perform the dimensional reduction and obtain explicit forms of the type IIA fields.

## 2. Manifolds with $G_2$ and Spin(7) holonomy from quaternionic spaces

Consider M-theory on the manifold $M_4 \times X_7$ where $M_4$ is the flat 4-d Minkowski space. The resulting 4-d field theory exhibits $\mathcal{N}$=1 supersymmetry if $X_7$ allows for exactly one (covariantly constant) Killing spinor and in the absence of $G$-fluxes this is the case if the manifold $X_7$ has $G_2$ holonomy. The exceptional group $G_2$ appears as automorphism group of octonions: $o = x^0\mathbb{I} + x^a i_a$, where $i_a$ satisfy the algebra

$$i_a i_b = -\delta_{ab} + \psi_{abc}\, i_c\,,$$

and the $G_2$-invariant 3-index tensor $\psi_{abc}$ is given in the standard basis by

$$\begin{aligned}\Psi &= \frac{1}{3!}\psi_{abc}\, e^a \wedge e^b \wedge e^c \\ &= e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^3 \wedge e^5 + e^5 \wedge e^1 \wedge e^6 + e^6 \wedge e^2 \wedge e^4 + \\ &\quad + e^4 \wedge e^7 \wedge e^1 + e^5 \wedge e^7 \wedge e^2 + e^6 \wedge e^7 \wedge e^3\,, \\ &= e^1 \wedge e^2 \wedge e^3 + \frac{1}{2}\, e^i \wedge e^m \wedge J^i_{mn} e^n \end{aligned} \tag{2.1}$$

where $J^i_{mn}$ ($i = 1, 2, 3,\ m = 4, 5, 6, 7$) are the anti-selfdual ($J^i_{mn} = -\frac{1}{2}\epsilon_{mnpq}\, J^i_{pq}$) complex structures defined by the algebra

$$J^i \cdot J^j = -\mathbb{1}\, \delta^{ij} + \epsilon^{ijk} J^k\,. \tag{2.2}$$

$G_2$-holonomy requires that this 3-index tensor is closed and co-closed

$$d\Psi = d^\star \Psi = 0 \tag{2.3}$$

which implies that $\Psi$ is a covariantly constant 3-form and is equivalent to the existence of a Killing spinor. This in turn is ensured if the spin connection satisfies the projector condition [37, 38]

$$\psi_{abc}\, \hat{\omega}^{bc} = 0\,. \tag{2.4}$$

Both conditions (2.3) and (2.4) yield a set of first order differential equations for the metric functions. If the manifold allows for more covariantly constant form-fields, the holonomy is further restricted and the Killing spinor equation has more than one solution so that the 4-dimensional model has extended supersymmetry.

As it has been shown in [29, 30] (see also [15] where our notations are used) a metric that fulfills these equations is given by

$$ds^2 = \frac{1}{\sqrt{2\kappa|u|^2 + u_0}}\,(du^i + \epsilon^{ijk} A^j u^k)^2 + \sqrt{2\kappa|u|^2 + u_0}\; ds_4^2\,. \tag{2.5}$$

which is topologically a $\mathbb{R}^3$ bundle (related to the coordinates $u^i$) over a quaternionic base space, given by the metric $ds_4^2$ with the curvature $\kappa$ and the SU(2) connection $A^i$ ($u_0$ is an integration constant); see next section for our conventions. This $G_2$ metric is, up to SU(2) rotations of the complex structures, fixed by the quaternionic base space and in the next section we discuss in detail the quaternionic space that we want to consider. In the



JHEP08(2002)027

limit $\kappa = 0$ this space becomes hyper-Kähler with vanishing SU(2) curvature and hence the connection $A^i$ gives a pure gauge transformation, see eqs. (3.2) and (3.4). Therefore, the connection part in (2.5) can be absorbed by a proper SU(2) rotation of the $u^i$ coordinates and the space becomes a direct product of $\mathbb{R}^3$ and the hyper Kähler space. But also if the curvature is non-trivial, there is still the freedom to choose a proper SU(2) basis.

For $\kappa \neq 0$ we can also introduce polar coordinates for the $\mathbb{R}^3$ part and the metric becomes

$$ds^2 = \frac{dr^2}{\kappa(1 - 4u_0/r^4)} + \frac{r^2}{4\kappa}\left(1 - \frac{4u_0}{r^4}\right) g_{ab}\left(dx^a + \xi^a_i A^i\right)\left(dx^b + \xi^b_j A^j\right) + \frac{r^2}{2} ds_4^2 \,, \tag{2.6}$$

where $g_{ab}$ is the metric of $\mathbb{S}^2$ with the three Killing vectors $\xi^a_i$. In the limit $u_0 \to 0$ this metric is a cone over a 6-manifold $Y$ which is a $\mathbb{S}^2$ bundle over the quaternionic space $Q$ and this manifold has a weak SU(3) holonomy. To see this we write the 7-metric (with $u_0 = 0$ and for $\kappa = 1$) as

$$ds^2 = dr^2 + r^2 ds_Y^2 \,. \tag{2.7}$$

Decomposing the fibered $\mathbb{R}^3$ as

$$\begin{aligned} u^1 &= |u| \cos\theta \,, \\ u^2 &= |u| \sin\theta\, \cos\varphi \,, \\ u^3 &= |u| \sin\theta\, \sin\varphi \,, \end{aligned} \tag{2.8}$$

the metric of the six-dimensional base becomes

$$ds_Y^2 = V^a \otimes V^a \,, \tag{2.9}$$

where

$$\begin{aligned} V^1 &= \frac{\hat{e}^1}{r} \equiv \frac{1}{2}\left(d\theta - \sin\varphi A^2 + \cos\varphi A^3\right) , \\ V^2 &= \frac{\hat{e}^2}{r} \equiv \frac{1}{2}\left(\sin\theta\, d\varphi + \sin\theta A^1 - \cos\theta\cos\varphi A^2 - \cos\theta \sin\varphi A^3\right) , \\ V^m &= \sqrt{\frac{\kappa}{2}}\, e_4^m \end{aligned} \tag{2.10}$$

(where $e_4^m$ is the vielbein of the quaternionic space). We can now show that this manifold is half-flat, which, according to [39], implies a reduction to SU(3) defined by $\omega$ and $\psi_\pm$ for which $\hat{d}\psi_+ = 0$ and $\omega \wedge \hat{d}\omega = 0$, but $\hat{d}\omega \neq 0$ (where the differential $\hat{d}$ is taken on the six-dimensional subspace). This implies that $Y$ has weak SU(3) holonomy, as it is expected. From the SU(3) forms one can build the harmonic 3-form $\Psi$ which defines the $G_2$ structure as

$$\Psi = \omega \wedge dr + \psi_+ \,. \tag{2.11}$$

In our case the two-form $\omega$ is given by

$$\omega \equiv \hat{e}^1\hat{e}^2 + \hat{e}^3\hat{e}^4 + \hat{e}^5\hat{e}^6 \,, \tag{2.12}$$

and the three-form $\psi_+$ satisfies

$$\psi_+ \equiv \frac{1}{3} d\omega = \hat{e}^1\hat{e}^3\hat{e}^5 - \hat{e}^1\hat{e}^4\hat{e}^6 - \hat{e}^2\hat{e}^3\hat{e}^6 - \hat{e}^2\hat{e}^4\hat{e}^5 \,. \tag{2.13}$$

The SU(3) reduction is completed by another three-form $\psi_-$, defined such that they satisfy the compatibility relations $\omega \wedge \psi_\pm = 0$ and $\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3$. We have already explicitly constructed $\hat{e}^1$ and $\hat{e}^2$ in (2.10) and we can obtain the rest of the six-dimensional orthonormal base $\hat{e}^i$ performing a $\theta$ and $\varphi$ dependent SO(4) rotation of the seven-dimensional base $e^i$:

$$r\hat{e}^3 = \sin\theta\, e^4 + \cos\theta \left(\cos\varphi\, e^5 + \sin\varphi\, e^6\right) , \tag{2.14}$$
$$r\hat{e}^3 = \cos\varphi\, e^6 - \sin\varphi\, e^5 \,, \tag{2.15}$$
$$r\hat{e}^5 = -e^7 \,, \tag{2.16}$$
$$r\hat{e}^6 = -\cos\theta\, e^4 + \sin\theta \left(\cos\varphi\, e^5 + \sin\varphi\, e^6\right) . \tag{2.17}$$

Let us end this section with a comment on 8-manifolds with Spin(7) holonomy. Again, they allow for one (covariantly constant) Killing spinor and yield therefore $\mathcal{N}$=1 supersymmetry in three dimension upon dimensional reduction. The construction is again fixed by a 4-d quaternionic space $Q$ and the metric reads [29, 30] (see also [40, 9] for generalizations)

$$ds^2 = \frac{dr^2}{\kappa(1 - u_0/r^{10/3})} + \frac{9}{100\,\kappa} r^2 \left(1 - \frac{u_0}{r^{10/3}}\right) \left(\sigma^i - A^i\right)^2 + \frac{9}{20} r^2\, ds_4^2 \tag{2.18}$$

where $u_0$ is again an integration constant and $\sigma^i$ are the left-invariant one-forms on SU(2). Topologically, this space is an $\mathbb{R}^4$ bundle over the quaternionic space and the cone $Y$ (orbits of constant $r$) is now an $\mathbb{S}^3$ bundle over $Q$.

## 3. Quaternionic space with two commuting isometries

In the last section we have introduced the class of manifolds with $G_2$ and Spin(7) holonomy, which are basically fixed by a quaternionic base space. In this section we will consider a specific quaternionic space with two isometries that we later-on want to employ for $G_2$ spaces.

### 3.1 General conventions

Quaternionic-Kähler spaces are complex spaces that allow for three complex structures $J^i$ ($i = 1, 2, 3$) defined by the algebra (2.2). Denoting the quaternionic vielbein by $e^m$, one obtains three 2-forms $\Omega^i$ by

$$\Omega^i = -\frac{\kappa}{2}\, e^m \wedge J^i_{mn} e^n \,. \tag{3.1}$$

The holonomy of a 4n-dimensional quaternionic spaces is contained in Sp($n$) × SU(2). This statement is trivial for $n = 1$ and can be replaced by the requirement that the Weyl-tensor of 4-dimensional quaternionic space has to be anti-selfdual

$$W + {}^\star W = 0 \,.$$

For a quaternionic space in any dimension the triplet of 2-forms $\Omega^i$ is expressed in terms of the SU(2)-part of the quaternionic connection $A^i$ as

$$dA^i + \frac{1}{2}\epsilon^{ijk}A^j \wedge A^k = \Omega^i \tag{3.2}$$

which ensures that the triplet of 2-forms is covariantly constant. Moreover, any quaternionic space is an Einstein space with curvature $\kappa$ implying that its metric $g_{mn}$ solves the equation

$$R_{mn} = 3\,\kappa\, g_{mn}\,. \tag{3.3}$$

The complex structures can be selfdual or anti-selfdual and in our notation we will take the latter ($J^i_{mn} = -\frac{1}{2}\epsilon_{mnpq}\, J^i_{pq}$) so that the triplet of 2-forms can be written as

$$\begin{aligned}
\Omega^1 &= -\kappa\left(e^4 \wedge e^7 - e^5 \wedge e^6\right), \\
\Omega^2 &= -\kappa\left(e^4 \wedge e^6 + e^5 \wedge e^7\right), \\
\Omega^3 &= -\kappa\left(-e^4 \wedge e^5 + e^6 \wedge e^7\right).
\end{aligned} \tag{3.4}$$

Moreover, the SU(2) connection is given as the anti-selfdual part of the spin connection $\omega^{mn}$ of the quaternionic space

$$A^i = \frac{1}{2}\omega^{mn} J^i_{mn}\,. \tag{3.5}$$

In the same way, the selfdual part gives the Sp($n$) connection.

### 3.2 Deriving the explicit metric

The maximally symmetric 4d quaternionic space has 10 isometries spanning a group of rank two (SO(5) or SO(4, 1)) and hence there are at most two commuting isometries. We are interested in the situation, where the space admits only these two isometries and all others are broken. This can be done by a double orbifold, which imposes non-trivial periodicities along these two directions. Hence, consider the metric ansatz

$$ds_4^2 = \frac{1}{F^2(p,q)}\left[\frac{dp^2}{P(p)} + P(p)\, d\tau^2 + \frac{dq^2}{Q(q)} + Q(q)\, d\sigma^2\right] \tag{3.6}$$

where $\partial_\tau$ and $\partial_\sigma$ are the two commuting Killing vectors and (single) zeros of $P$ and $Q$ require non-trivial periodicity in $\tau$ and $\sigma$. Since the metric has to be Einstein, we can derive the function $F(p,q)$ from the combination of the Ricci tensor

$$0 = R_p{}^p - R_\tau{}^\tau = 2F\,\partial_p^2 F\,, \qquad 0 = R_q{}^q - R_\sigma{}^\sigma = 2F\partial_q^2 F\,.$$

Taking as solution $F = p+q$ and calculating another combination of the Ricci tensor yields

$$0 = \partial_p\partial_q\left(\frac{R_\sigma{}^\sigma - R_\tau{}^\tau}{p+q}\right) = \frac{1}{2}\left[Q'''(q) - P'''(p)\right]$$

and therefore $P$ are $Q$ are polynomials of third degree. It is straightforward to investigate the other equations and one finds as general solution of the equation (3.3): $P = a_0 - \kappa + a_1 p + a_2 p^2 + a_3 p^3$, $Q = -a_0 + a_1 q - a_2 q^2 + a_3 q^3$. The Weyl tensor for this space is anti-selfdual only if: $a_3 = 0$. So, this quaternionic space depends in total on four parameters that fix the identifications for $\sigma$ and $\tau$. The torus spanned by these two isometries is diagonal, but one can also deform the torus while keeping the quaternionic property. Fortunately, the corresponding metric has been known for quite some time. It was introduced as Minkowskean solution by Demianski and Plebanski [35, 36] and a discussion in the mathematical literature is given e.g. in [41, 42, 43], see also [44, 45, 16, 17] for more general quaternionic spaces with two isometries. The corresponding euclidean metric reads

$$ds_4^2 = \frac{1}{(1+pq)^2}\left[\frac{p^2-q^2}{P}dp^2 + \frac{p^2-q^2}{Q}dq^2 + \frac{P}{p^2-q^2}\left(d\tau + q^2 d\sigma\right)^2 + \frac{Q}{p^2-q^2}\left(d\tau + p^2 d\sigma\right)^2\right] \tag{3.7}$$

where the polynomials are now given by $P = \alpha - 2np - \epsilon p^2 + 2mp^3 + (\alpha - \kappa)p^4$, $Q = -\alpha + 2mq + \epsilon q^2 - 2nq^3 - (\alpha - \kappa)q^4$ and the Weyl tensor becomes anti-selfdual iff: $m = n$. Again (single) zeros of $P$ and $Q$ are conical singularities, which gives periodicities of $\sigma$ and $\tau$ defining the deformed torus. In order to recover the form (3.6), one makes the transformation $p \to 1/p$ combined with $(p, q, \tau, \sigma) \to \frac{1}{\lambda}(p, q, \tau, \sigma)$ and $(\alpha, n, \epsilon, m, \kappa) \to (\alpha, \lambda n, \lambda^2\epsilon, \kappa)$ followed by the limit $\lambda \to \infty$, see also [35, 36].

However, we do not want to use this form of the metric and apply another scaling: $(p, q) \to \lambda(p, q)$, $(\alpha, n, \epsilon, m, \kappa) \to (\alpha\lambda^4, n\lambda^3, \epsilon\lambda^2, m\lambda^3, \kappa)$ followed by the limit $\lambda \to 0$. As a consequence the metric becomes

$$ds_4^2 = \frac{p^2-q^2}{P}dp^2 + \frac{p^2-q^2}{Q}dq^2 + \frac{P}{p^2-q^2}\left(d\tau + q^2 d\sigma\right)^2 + \frac{Q}{p^2-q^2}\left(d\tau + p^2 d\sigma\right)^2 \tag{3.8}$$

with

$$P = \alpha - 2np - \epsilon p^2 - \kappa p^4\,, \qquad Q = -\alpha + 2mq + \epsilon q^2 + \kappa q^4\,. \tag{3.9}$$

In the following we will use this form of the metric, which has again an anti-selfdual Weyl tensor iff: $m = n$. In this case the two polynomials become, up to the overall sign, identical and we can use the notation

$$\begin{aligned} P &= -\kappa(p-r_1)(p-r_2)(p-r_3)(p-r_4)\,, \\ Q &= \kappa(q-r_1)(q-r_2)(q-r_3)(q-r_4)\,, \\ 0 &= r_1 + r_2 + r_3 + r_4\,. \end{aligned} \tag{3.10}$$

In addition to the two abelian isometries, there are the following symmetries

$$\begin{aligned} &(i) \quad p \leftrightarrow q\,, \\ &(ii) \quad p \to -p\,, \qquad q \to -q\,, \qquad r_i \to -r_i\,, \\ &(iii) \quad (p, q, \tau, \sigma) \to \left(\lambda\, p, \lambda\, q, \frac{\tau}{\lambda}, \frac{\sigma}{\lambda^3}\right) \qquad \text{and} \quad r_i \to \lambda\, r_i\,. \end{aligned} \tag{3.11}$$

The last symmetry can be used to scale one non-vanishing parameter to $\pm 1$. We have therefore the following interpretation of the parameters: one is obviously the cosmological constant, two parameterize the orbifolds and turning off one of them yields an enhancement



JHEP08(2002)027

of one U(1) isometry to one of three different groups SU(2), $\mathrm{SL}(2,R)$ or the Heisenberg group, that are related to the three discrete values of the fourth parameter (see also next subsection).

It is important to note, that the physical parameter range is given by the values of $(p,q)$ which fulfill the two inequalities

$$(p^2-q^2)\,P(p) \geq 0 \qquad \text{and} \quad (p^2-q^2)\,Q(q) \geq 0\,. \tag{3.12}$$

This allows for a number of different coordinate regions, which are separated by regions that contain two timelike coordinates. Note, these timelike regions appear beyond fixed points of the isometries, which become branes upon dimensional reduction. Thus, they indicate the appearance of additional massless modes and should be interpreted as phase transition points.

An important property of this space is the presence of a curvature singularity, which becomes visible in the square of the Riemann curvature

$$R_{abcd}R^{abcd} = 24\kappa^2 + \frac{96\,n^2}{(p+q)^6} \tag{3.13}$$

where $n$ was the coefficient of the linear part in the polynomials. This co-dimension one singularity at $p+q=0$ is present for any value of the fiber coordinates $u^i$ and hence is a singularity also of the 7-manifold (actually, it is singular domain wall of the whole 11-dimensional space time). There are two limits in which this curvature singularity disappears. One is obviously given by $n=0$ and the other by $n\to\infty$ combined with a proper rescaling of $p$ and $q$. As we will discuss in the next section, both limits yield a homogeneous quaternionic space; $\mathbb{S}^4$ or $\mathbb{CP}^2$ (or their non-compact versions).

Having the metric it is straightforward to determine the SU(2) connections as introduced in (3.5). They are given by

$$\begin{aligned}
A^1 &= \frac{\sqrt{PQ}}{(q-p)}\,d\sigma\,,\\
A^2 &= -\kappa(p-q)\,d\tau + \frac{1}{(p-q)}\left[\alpha - n(p+q) - \epsilon\,qp - \kappa p^2q^2\right]\,d\sigma\\
A^3 &= \frac{1}{(p-q)}\left[\sqrt{\frac{Q}{P}}\,dp + \sqrt{\frac{P}{Q}}\,dq\right].
\end{aligned} \tag{3.14}$$

and fulfill the relations (3.2) and (3.4) with

$$\begin{aligned}
e^4 &= \sqrt{p^2-q^2}\,\frac{dp}{\sqrt{P}}\,, & e^5 &= -\sqrt{p^2-q^2}\,\frac{dq}{\sqrt{Q}}\,,\\
e^6 &= \frac{\sqrt{P}}{\sqrt{p^2-q^2}}\left(d\tau + q^2 d\sigma\right), & e^7 &= -\frac{\sqrt{Q}}{\sqrt{p^2-q^2}}\left(d\tau + p^2 d\sigma\right).
\end{aligned} \tag{3.15}$$

### 3.3 Special limits

The 4-dimensional base space as introduced in the last subsection, can be obtained by a Wick rotation of a solution that has been discussed by Plebanski and Demianski as a "Rotating, Charged, and Uniformly Accerating Mass in General Relativity" [37]. It is also



JHEP08(2002)027

known as the (A)dS-Kerr-Newman-Taub-NUT solution, where the electric and magnetic charges are obviously zero in our application. To make the relation to these known Einstein spaces more clear, let us perform the corresponding limits.

To obtain the euclidean (A)dS-Kerr-Newman-Taub-NUT solution as a limit of our euclidean PD solution, we set (see also [46])

$$\begin{aligned}
&q = r\,, \qquad p = a\cos\theta + N\,, \qquad \tau = t + \left(\frac{N^2}{a} + a\right)\frac{\phi}{\Xi}\,, \qquad \sigma = -\frac{\phi}{a\Xi}\,,\\
&\alpha = -a^2 + N^2\left(1 - \kappa\, 3a^2 + \kappa\, 3N^2\right),\\
&n = N\left[1 - \kappa\, a^2 + 4\kappa N^2\right],\\
&\epsilon = -1 - \kappa\, a^2 - 6\kappa N^2\,,\\
&\Xi = 1 - \kappa a^2\,.
\end{aligned} \tag{3.16}$$

With these transformations and relaxing the constraint $m = n$ (so that the Weyl tensor is not anti-selfdual), the polynomials $P$ and $Q$ become

$$P = -a^2\sin^2\theta\left[1 - \kappa\left(4aN\cos\theta + a^2\cos^2\theta\right)\right], \tag{3.17}$$

$$Q = -(r^2 + N^2) + \kappa\left(r^4 - a^2r^2 - 6N^2r^2 + 3a^2N^2 - 3N^4\right) + 2mr + a^2\,. \tag{3.18}$$

If we moreover define,

$$R^2 = r^2 - (a\cos\theta + N)^2\,, \tag{3.19}$$

$$\lambda = (r^2 + N^2) - \kappa\left(r^4 - a^2r^2 - 6N^2r^2 + 3a^2N^2 - 3N^4\right) - 2mr - a^2 \tag{3.20}$$

one gets,

$$\begin{aligned}
\frac{p^2-q^2}{Q(q)}dq^2 &= \frac{R^2}{\lambda}dr^2\,,\\
\frac{p^2-q^2}{P(p)}dp^2 &= \frac{R^2}{1-\kappa(a^2\cos^2\theta + 4aN\cos\theta)}d\theta^2\,,\\
\frac{Q(q)}{p^2-q^2}\left(d\tau + p^2 d\sigma\right)^2 &= \frac{\lambda}{R^2}\left[dt + \frac{a\sin^2\theta - 2N\cos\theta}{\Xi}d\phi\right]^2\\
\frac{P(p)}{p^2-q^2}\left(d\tau + q^2 d\sigma\right)^2 &= \frac{\sin^2\theta\left[1 - \kappa a\cos\theta(a\cos\theta + 4N)\right]}{R^2}\left[adt - \frac{(r^2-a^2-N^2)}{\Xi}d\phi\right]^2
\end{aligned}$$

and we obtain the euclidean (A)dS Kerr-Taub-NUT solution given by

$$\begin{aligned}
ds_4^2 = {}& \frac{R^2}{1-\kappa(a^2\cos^2\theta + 4aN\cos\theta)}d\theta^2 + \frac{R^2}{\lambda}dr^2 +\\
&+ \frac{\lambda}{R^2}\left[dt + \left(\frac{a\sin^2\theta}{\Xi} - \frac{2N\cos\theta}{\Xi}\right)d\phi\right]^2 +\\
&+ \frac{\sin^2\theta\left[1 - \kappa(a^2\cos^2\theta + 4aN\cos\theta)\right]}{R^2}\left[adt - \frac{(r^2-a^2-N^2)}{\Xi}d\phi\right]^2.
\end{aligned} \tag{3.21}$$

The limits are now straightforward: if $N = 0$ one obtains the euclidean (A)dS-Kerr solution, where $a$ corresponds to the rotational parameter. But note, there is no rotation in an euclidean space, the axial symmetric minkowskean Kerr-solution becomes instead an euclidean dipole solution. In fact, the euclidean Kerr solution (i.e. for $\kappa = 0$) has been identified in [47] as Taub-NUT/anti-Taub-NUT dipole solution where the parameter $a$ just measures the distance between the two centers. On the other hand, if $a = 0$ while $N \neq 0$, $\kappa \neq 0$, the solution becomes euclidean (A)dS-Taub-NUT given by [48]

$$ds_4^2 = V(r)\,(dt - 2N\cos\theta d\phi)^2 + \frac{dr^2}{V(r)} + (r^2 - N^2)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.22}$$

where

$$V(r) \equiv \frac{\lambda}{R^2} = \frac{1}{r^2 - N^2}\left[(r^2 + N^2) - \kappa\left(r^4 - 6N^2r^2 - 3N^4\right) - 2mr\right]. \tag{3.23}$$

The isometry group for this space has been enhanced to U(1) × SU(2) (from U(1) × U(1) for $a \neq 0$) and the relevance of this quaternionic space in gauge supergravity and for $G_2$ manifolds has been discussed recently in [49, 15]. In the limit of vanishing $N$, the space becomes euclidean $(A)dS_4$ (i.e. $\mathbb{S}^4$ or the non-compact hyperboloid), which is maximal symmetric with 10 isometries parameterizing SO(5) or SO(4, 1). On the other hand, in the limit $N \to \infty$ while keeping $\hat{r} = N(r - N)$ fix, the solution becomes the coset space SU(3)/ U(2) (= $\mathbb{CP}^2$) or SU(2, 1)/ U(2) resp. This is the second known regular 4-dimensional quaternionic space, which has 8 isometries parameterizing SU(3) or SU(2, 1). It is also instructive to understand these limits in terms of the four roots $r_m$ as introduced in (3.10). The maximal symmetric spaces ($\mathbb{S}^4$ resp. $EAdS_4$) can be obtained if

$$r_1 = -r_4\,, \qquad r_2 = -r_3 \tag{3.24}$$

and the corresponding transformation is given in [37] (for Minkowskean signature). On the other hand, for $N \to \infty$ we find from (3.16): $\alpha = 3\kappa N^4$, $n = 4\kappa N^3$, $\epsilon = -6\kappa N^2$ yielding: $P(p) = -\kappa(-3N^4 + 8N^3p - 6N^2p^2 + p^4) = \kappa(N - p)^3(p + 3N)$. Thus, one gets $\mathbb{CP}^2$ or its non-compact analog in the limit where three zeros of the polynomial coincide, as e.g.

$$r_2 = r_3 = r_4 = N \qquad \text{and} \qquad N \to \infty\,. \tag{3.25}$$

This limit is of course only regular if one shifts also $q$ and $p$ (see eqs. (3.16) and recall the replacement $r = \hat{r}/N + N$, see also [49, 15]).

## 4. Fixed point set

### 4.1 General discussion

The quaternionic space has two Killing vectors and let us consider the isometry obtained by the linear combination

$$k = \beta_1\partial_\tau - \beta_2\partial_\sigma\,. \tag{4.1}$$

Since the SU(2) connection $A^i$ does not depend on $\sigma$ and $\tau$, this Killing vector corresponds to an isometry also of the $G_2$ manifold (2.6). To find the fixed points of such isometry, we have to satisfy the equations $|k|^2 = 0$ and with the metric (2.6) we find

$$
\begin{aligned}
|k|^2 &= \frac{r^2}{4\kappa}\left(1-\frac{4u_0}{r^4}\right) g_{ab}\xi_i^a\xi_i^b\,(\beta_1 A_\tau^i - \beta_2 A_\sigma^i)^2 + \\
&+ \frac{r^2}{2}\left[\frac{P}{p^2-q^2}\left(\beta_1-\beta_2 q^2\right)^2 + \frac{Q}{p^2-q^2}\left(\beta_1-\beta_2 p^2\right)^2\right] = 0\,.
\end{aligned} \tag{4.2}
$$

This is one necessary condition on fixed points, but in order to ensure that it is at finite geodesic distance one has to require that the fixed point set is non-degenerate, i.e. $(\nabla k)^2 \neq 0$ at $|k|^2 = 0$. If this condition is not fulfilled the space exhibits a infinite throat and the fixed point will be at infinity (gauging such an isometry in gauged supergravity results in a run-away solution, see [49]). Actually in this case the Killing vector does not parameterize a rotational symmetry, but a translational one. This happens e.g., if the fourth order polynomials have a double zero, see also the discussion below.

Since the physical parameter range of the $(p,q)$ coordinates is given by the values which fulfill the inequalities

$$
\left(p^2-q^2\right) P(p) \geq 0 \qquad \text{and} \qquad \left(p^2-q^2\right) Q(q) \geq 0\,, \tag{4.3}
$$

each term in (4.2) has to vanish separately. Apart from the trivial zero at $r = 0$, the second term of $|k|^2$ vanishes for the two cases

$$
\begin{aligned}
&(a) \quad P = 0 \qquad \text{and} \quad Q = 0\,, \\
&(b) \quad p = \sqrt{\frac{\beta_1}{\beta_2}} = r_m \qquad \text{or} \quad q = \sqrt{\frac{\beta_1}{\beta_2}} = r_m
\end{aligned} \tag{4.4}
$$

where $r_m$ is one of the roots of $P(p)$ [respectively $Q(q)$]. The condition $(a)$ fixes $p$ and $q$ at points where the two isometric U(1) fibers in the metric vanish and hence this condition defines a point on the quaternionic space and is called a NUT. On the other hand, condition $(b)$ fixes only one coordinate ($p$ or $q$) and only one U(1) fiber vanishes and therefore this condition defines a 2-dimensional subspace — a bolt. Obviously this latter case can only happen for a specific Killing vector, a generic choice of $\beta_1$ and $\beta_2$ will not yield bolts. For both cases $(a)$ and $(b)$ only $A^2_\sigma$ and $A^2_\tau$ are non-trivial and hence the first term in (4.2) is zero iff

$$
\begin{aligned}
&(c) \qquad |\xi_2|^2 = 0\,, \\
&(d) \qquad A^2_\mu\, k^\mu = \beta_1 A^2_\tau - \beta_2 A^2_\sigma = 0 \quad \text{or} \\
&(e) \qquad\quad r^4 = 4u_0\,.
\end{aligned} \tag{4.5}
$$

The last case is only of interest as long as $u_0 \neq 0$ and corresponds to the point where the $\mathbb{S}^2$ has collapsed to a point while the quaternionic space is still finite. Case $(c)$ is satisfied at fixed points of the second $\mathbb{S}^2$-Killing vector (i.e. $|\xi_2|^2 = 0$) and this gives exactly two (antipodal) points on $\mathbb{S}^2$, which in the coordinates (2.8) are given by $\cos\theta = \sin\varphi = 0$ (or $u^1 = u^3 = 0$). For case $(d)$ one finds

$$
\beta_1 A^2_\tau - \beta_2 A^2_\sigma = \frac{(\beta_1-\beta_2 q^2)[(p-q)\partial_p - 2]P - (\beta_1-\beta_2 p^2)[(p-q)\partial_q + 2]Q}{2(p-q)^2(p+q)} \tag{4.6}
$$

and this has to vanish in combination with case $(a)$ or $(b)$. By inserting the polynomials (3.10) one finds, for generic values of $\beta_1$ and $\beta_2$, that this can only happen at double zeros of $P$ or $Q$. But as we discussed before these double zeros correspond to degenerate fixed points which are not at finite geodesic distance. On the other hand, in combination with case $(a)$, we find always a ratio of $\frac{\beta_1}{\beta_2} = \beta$ for which this combination vanishes at zeros of $Q$ and $P$, see also the explicit example below. Notice, for these simple Killing vectors the combination (4.6) gives the Killing prepotential (momentum maps) and for a 4-dimensional quaternionic space with at least two abelian compact Killing vectors there is exactly one combination for which the Killing prepotentials (or momentum maps) vanish at the fixed point [50, 51].

In summary, depending on the choice of parameters there are fixed point sets of various co-dimensions:

**Fixed point set of co-dimension 7** These are zeros of $|k|^2$ which are points on the 7-manifold. This is the case at the conical singularity at $r = 0$ if $u_0 = 0$ or otherwise a combination of the constraint $(a)$ with $(e)$.

**Fixed point set of co-dimension 6** They are related to a combination of case $(a)$ and $(c)$, which means that the fixed point set is given by a NUT on the quaternionic space combined with a fixed point of the second $\mathbb{S}^2$ Killing vector. Since we have two abelian isometries we can first reduce over the $k$ to get a IIA configuration followed by a T-duality over the second isometry. In this procedure, these co-dimension 6 fixed points should be mapped onto type IIB NS5-branes, because they are fixed points of both isometries of the 7-manifold and hence are also fixed points of translations along the T-duality direction.

**Fixed point set of co-dimension 5** They are only present if $u_0 \neq 0$ and correspond to a combination of case $(b)$ and $(e)$, but they are not additional isolated fixed point sets. In fact, $r^4 = 4u_0$ represents exactly the point of minimal distance of given codimension 4 fixed points set. The same is true for the codimension 7 fixed point appearing as combination of case $(a)$ and $(e)$, which is the orbit of minimal distance between given co-dimension 6 fixed point sets.

**Fixed point set of co-dimension 4** These are perhaps the most interesting ones, since they are identified as 6-branes upon the reduction to type IIA string theory. We obtain co-dimension 4 fixed points as a combination of case $(b)$ and $(c)$ as well as of case $(a)$ and $(d)$. In both situations the 6-branes will wrap a 2-cycle of the 6-manifold $Y$: for the combination $(b)$ and $(c)$ this 2-cycle is the bolt inside the quaternionic space and if $p$ and $q$ are bounded by two roots $r_m$, this 2-cycle is topologically a line segment times a circle and if there are no conical singularities this 2-cycle becomes topologically an $\mathbb{S}^2$. Recall, case $(b)$ as well as case $(d)$ require specific Killing vectors which *do not* agree, but in any case 6-branes appear only for a proper choice of the $11th$ coordinate. For the combination $(a)$ and $(d)$, the 6-branes are transversal to the quaternionic space and wrap all three $u^i$ coordinates.



JHEP08(2002)027

**Fixed point set of co-dimension 2** These fixed points can appear only as a combination of case ($b$) and case ($d$). As we mentioned after equation (4.6) this requires that $p$ or $q$ run toward a double zero of $P$ or $Q$ and from the metric (3.8) we see that these double zeros are not at finite geodesic distance. Instead, near these points the space develops a throat and the fixed point set is at infinity and we can discard them. Alternatively, case ($d$) can appear for a specific Killing vector, which is however different from the one fixed by case ($b$) and hence there are no co-dimension 2 fixed points.

Recall, in addition to these fixed points there is the co-dimension one curvature singularity of the quaternionic space at $p + q = 0$.

### 4.2 Nuts and bolts of the quaternionic space

In order to determine the number of fixed points, we have to ask for the number of solutions of the equations (4.4), which are related to NUTs and bolts on the quaternionic space. Given that our polynomial $P$ or $Q$ has four roots $r_m$ one can distinguish among four main cases:

$$
\begin{aligned}
&i) \quad \kappa > 0 \text{ while } r_2, r_3, r_4 \geq 0 \text{ and } r_1 < 0\,,\\
&ii) \quad \kappa > 0 \text{ while } r_3, r_4 \geq 0\,,\ r_1, r_2 \leq 0 \text{ and } r_4 + r_1 > 0\,,\\
&iii) \quad \kappa < 0 \text{ while } r_2, r_3, r_4 \geq 0 \text{ and } r_1 < 0\,,\\
&iv) \quad \kappa < 0 \text{ while } r_3, r_4 \geq 0\,,\ r_1, r_2 \leq 0 \text{ and } r_4 + r_1 \geq 0\,.
\end{aligned}
\tag{4.7}
$$

Every other case can be reconducted to one of the above upon using some of the symmetries (3.11) of the metric (3.8).

Let us start with the discussion of the possible bolts. Any of the four roots for which $\sqrt{\beta_1/\beta_2} = r_m$ gives bolts and since $p$ and $q$ can go independently to this root there are always two bolts. But note, not each coordinate region contains a bolt. E.g. if $p$ is in region IV (see figure) and $q$ in region III we have two bolts if $r_m = \sqrt{\beta_1/\beta_2} = r_3$. On the other hand, if $\sqrt{\beta_1/\beta_2} = r_1$ one finds bolts only if one takes into account the other allowed coordinate regions, namely that $p$ is in region II and $q$ in region III or vice versa.

The discussion of all possible NUTs is more involved. It can be shown that one finds six solutions (less if some equality bounds are satisfied or if there are double roots) for any of the above possibilities in (4.7). They are never grouped in more than three in the same connected physical region of parameters. Actually one can find the following patterns: zero, one or three fixed points if the region does not contain the $p = -q$ singularity, two fixed points when the region contains the $p = -q$ singularity. We give here a table summarizing such possibilities, where we grouped the fixed points according to the $(p, q)$ sector they belong. For the cases with $\kappa > 0$ we find the fixed points summarized in table 1 whereas $\kappa < 0$ gives the fixed points summarized in table 2

Note, in the degenerate case where two or three roots are equal, one looses physical regions, which were defined by the relations (4.3) and recall, at double zeros the space develops a throat and effectively cuts the space in two disconnected regions. On the other hand, if $p$ and $q$ approach a single zeros from opposite sites this point is regular and one can pass this point.

| | | | |
|---|---|---|---|
| $(i)$ | $(r_4, r_1)$ | $(r_2, r_3)$ , $(r_2, r_4)$ , $(r_3, r_4)$ | $(r_2, r_1)$ , $(r_3, r_1)$ |
| $(ii)$ | $(r_3, r_4)$ , $(r_2, r_4)$ | $(r_2, r_1)$ , $(r_3, r_1)$ , $(r_3, r_2)$ | $(r_1, r_4)$ |

**Table 1:** These are all values of $(p, q)$ that are NUT fixed points of the quaternionic space with the parameters defined in (4.7). The fixed points in the same group are in the same physical region of parameters.

| | | | | | |
|---|---|---|---|---|---|
| $(iii)$ | $(r_4, r_3)$ | $(r_4, r_2)$ | $(r_3, r_2)$ | $(r_1, r_3)$ , $(r_1, r_4)$ | $(r_1, r_2)$ |
| $(iv)$ | $(r_4, r_3)$ | $(r_4, r_2)$ , $(r_4, r_1)$ | $(r_2, r_3)$ | $(r_1, r_3)$ | $(r_1, r_2)$ |

**Table 2:** These are the analogous fixed points for a negatively curved quaternionic space.

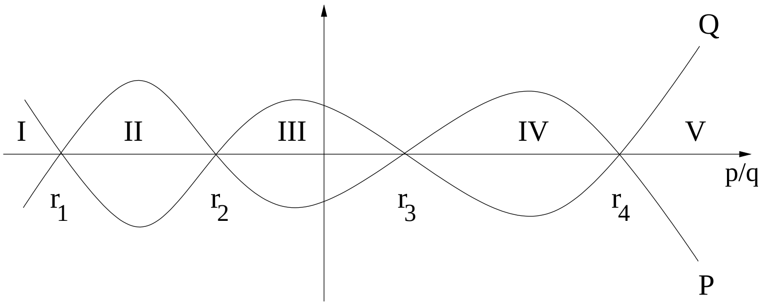


**Figure 1:** The quaternionic space is basically defined by two fourth order polynomial $P(p)$ and $Q(q)$ which differ only by a total sign. In this figure we have shown the case $\kappa > 0$ and denoted with $I, II, \ldots, V$ the different coordinate regions. The case with negative $\kappa$ corresponds effectively to an exchange of $P$ and $Q$.

### 4.3 Explicit example

Now we want to describe an explicit example which has only co-dimension 4 fixed points that become D6-branes upon compactification. Since there will be no other fixed points, the number of D6-branes is related to the number of co-dimension 4 fixed points [7, 14, 9]. As for the $\mathbb{CP}^2$ case, we will find that the fixed point set has two components and hence there are in total three stacks of D6-branes. An interesting question is to determine the number of 6-branes located at each fixed point. For the standard 6-brane, this number is related to the periodicity in the Taub-NUT space that resolves the conical singularity and also here, the number of 6-branes should be related to the conical singularity appearing at the fixed point. The corresponding deficit angle is given by the surface gravity of the corresponding fixed point set. This is a well-known quantity discussed in black hole physics, which is defined by $(\nabla k)_i^2$ calculated on the fixed point set $\Gamma_i$. It can be shown that this quantity is constant over the fixed point set and it gives, multiplied with the area of the fixed point set, the contribution to the Noether charge related to the Killing symmetry, see [52, 53]. Upon compactification this Noether charge gives the D6-brane charge. Applied to black holes, the surface gravity is the Hawking temperature, which is nothing but the inverse periodicity that resolves the conical singularity. It is straightforward to show, that the

JHEP08(2002)027

Taub-NUT space with NUT charge $N$ has a surface gravity of $|\nabla k| \sim 1/N$ and therefore we will identify the number of 6-branes of the fixed point set $\Gamma_i$ by: $N_i \sim 1/|\nabla k|_i$.

Let us now come to the concrete example. Recall, in order to obtain co-dimension 4 fixed points one has to consider a particular Killing vector so that

i. condition $(b)$ in eq. (4.4) is satisfied or alternatively

ii. the expression $(d)$ in (4.5) vanishes at a zero of $P$ and $Q$.

A closer look on equation (4.6) shows that both cases can only happen at the same time at double zeros of the polynomial yielding degenerate fixed points. We will therefore consider both cases independently.

For case $(i)$ we consider the Killing vector

$$k = r_3^2\,\partial_\tau - \partial_\sigma \tag{4.8}$$

which means that $r_3^2 = \beta_1$ $(\beta_2 = 1)$ and without further restrictions we will assume that $r_3 > 0$. In this example we will consider $p$ in region IV and $q$ in III or vice versa (see figure). There are now two sets of 6-branes located at

$$\begin{aligned} D6_1: &\quad p = r_3\,, \qquad u_1 = u_3 = 0\,, \\ D6_2: &\quad q = r_3\,, \qquad u_1 = u_3 = 0\,. \end{aligned} \tag{4.9}$$

But by keeping generic values of the roots, there will be further codimension 6 fixed points at $q = r_2$, $p = r_4$, $u_1 = u_2 = 0$ and at $p = r_2$, $q = r_4$, $u_1 = u_2 = 0$. In order to avoid these fixed points we will set $r_1 = r_2$, which essentially moves these fixed points to infinity since the metric (3.8) develops an infinite throat at $p \to r_2 = r_1$. Calculating the surface gravity for the fixed point set given in eq. (4.9) gives

$$|\nabla k|_1 = |\nabla k|_2 = \frac{\kappa}{4}\,(3r_3 + r_4)^2(r_4 - r_3) \tag{4.10}$$

where we used the constraint $0 = r_1 + r_2 + r_3 + r_4 = 2r_2 + r_3 + r_4$. That both numbers coincide, is a consequence of the symmetry $p \leftrightarrow q$ of the metric.

Next, let us consider the 6-branes coming from case $(ii)$, where the Killing vector

$$k = \beta\,\partial_\tau - \partial_\sigma \tag{4.11}$$

was fixed so that eq. (4.6) vanishes at a zero of $P$ and $Q$. Recall, this corresponds to the U(1) isometry for which the Killing prepotentials vanish, see [50, 51]. To be concrete, we will consider the fixed point: $p = r_4$ and $q = r_3$ and find that (4.6) vanishes at this point if

$$\beta = \frac{\beta_1}{\beta_2} = \frac{1}{2}\,[r_3(r_2 - r_4) + r_2(r_2 + r_4)]\,. \tag{4.12}$$

This yields 6-branes located at

$$\begin{aligned} D6_1: &\quad p = r_4\,, \qquad q = r_3\,, \\ D6_2: &\quad p = r_3\,, \qquad q = r_4\,. \end{aligned} \tag{4.13}$$

Both are NUTs on the quaternionic space and all three $u^i$ coordinates are part of their worldvolume. In order to avoid additional co-dimension 6 fixed points if $p$ (or $q$) run toward $r_2$ (see figure), we set again: $r_1 = r_2$ and $p$ (or $q$) becomes a non-compact coordinate and can be identified as the overall radial coordinate for the 6-brane intersection. This time we find for the surface gravity

$$|\nabla k|_1 = |\nabla k|_2 = \kappa \frac{r_3 r_4}{r_3 + r_4}(2r_3 + r_4)(2r_4 + r_3)\,. \tag{4.14}$$

As we argued at the beginning, this surface gravity should be related to the number D6-branes of two of the three stacks of 6-branes, which should be a quantized number. As explained in [54, 9], a consistent U(1) action requires that the ratio of two eigenvalues of the 2-form $dk$ calculated at the fixed point set should be a rational number. For the case at hand, this gives the quantization condition

$$\frac{n}{m} = \frac{1 + \sqrt{1-\Delta}}{|\Delta|}\,, \qquad \Delta = \frac{2(r_3^2 - r_4^2)}{r_4^4 + r_3^4 + 2} \tag{4.15}$$

where $n$ and $m$ are relative prime integers. This condition ensures, that a tangent vector at the fixed point comes back to its own if we go once around the circle (the U(1) action of a Killing vector acts as a rotation in the tangential plane given by two rotational parameters for a co-dimension 4 fixed point set).

## 5. Type IIA reduction

We will discuss now the reduction of the previous example along the compact direction determined by the isometry:

$$\partial_z = \partial_\sigma - \beta \partial_\tau\,. \tag{5.1}$$

This generic reduction becomes relevant to the intersecting $D6$-branes setup explained in the previous section when $\beta = r_3^2$ or it satisfies (4.12). From this choice we can introduce two new coordinates $w$ and $z$

$$\sigma = z\,, \qquad \tau = -\beta\, z + w\,, \tag{5.2}$$

such that $z$ becomes the coordinate along which we perform the reduction, while $w$ completes the set of the remaining ten-dimensional ones

$$x^\mu = \{y^a, u^i, p, q, w\} \qquad (a = 0, \dots 3)\,, \quad (i = 1, 2, 3)\,. \tag{5.3}$$

Here $y^a$ parametrize $\mathbb{R}^{1,3}$, $u^i$ were introduced before and parametrize $\mathbb{R}^3$ and $p$, $q$ and $w$ are the surviving coordinates of the quaternionic part. Performing the reduction using the standard Kaluza-Klein ansatz followed by a conformal rescaling, one produces a 10-dimensional IIA bosonic background with non-vanishing dilaton $\phi$ and metric in the NS-NS sector, whereas only the one-form $C_\mu$ is turned on in the RR sector. The relation between the two metrics, which also fixes the dilaton dependence, is given by

$$ds^2_{(11)} = e^{-\frac{2}{3}\phi(x)}\, g_{\mu\nu} dx^\mu dx^\nu + e^{\frac{4}{3}\phi(x)}\, \left[dz + dx^\mu C_\mu(x)\right]^2\,. \tag{5.4}$$

It is interesting to point out that the dilaton and the one-form can be completely determined in terms of the Killing vector (5.1) upon which we reduce the metric. This is much simpler in this new coordinate system. The relations for the dilaton and the one-form are

$$e^{\frac{4}{3}\phi} = g_{zz} = |k|^2\,, \tag{5.5}$$

$$C = e^{-\frac{4}{3}\phi} g_{zM} dx^M = |k|^{-2}\, k_M dx^M\,. \tag{5.6}$$

A first consequence is that the component of the one-form along the Killing vector direction is fixed

$$\imath_k C \equiv C_M k^M = 1\,, \tag{5.7}$$

which implies that $C$ has the right number of independent components. As a further consequence, one can determine its field strength in terms of $k$ and its derivatives

$$F_{MN} = 2\partial_{[M} C_{N]} = 6\,\frac{k^S k_{[S} \nabla_M k_{N]}}{|k|^4}\,. \tag{5.8}$$

Since near a brane configuration we expect to have some flux in ten dimensions, this should show up in the integral of the two-form $F$ on the transversal two-cycle $\mathcal{C}$:

$$\int_{\mathcal{C}} F \neq 0\,. \tag{5.9}$$

From the relations above and the fact that this flux should be quantized, we expect that its number could be read from the eigenvalues of the $\nabla k$ matrix. This is also compatible with the picture given in the previous section, where the surface gravity was related to the number of $D6$-branes and this latter was also derived from the $dk$ two-form.

We can now proceed to give the explicit expression for the various ten-dimensional fields. To do so, it is useful to define the following quantity

$$\mathcal{A} = \imath_k A^2 = \frac{\alpha - n(p+q) - \epsilon\, pq - \kappa\, p^2 q^2 + \beta\kappa (p-q)^2}{p-q}\,, \tag{5.10}$$

whose vanishing is related to the appearance of the co-dimension four singularities for the NUT fixed points of the quaternionic manifold. As we will see, this quantity appears repeatedly in the following formulae and this let us simplify the structure of the dilaton and $C$-field equations. The dilaton is determined to be

$$\begin{aligned} e^{\frac{4}{3}\phi} = {} & \frac{1}{\sqrt{2\kappa |u|^2 + u_0}} \left[ 2u_1\, u_2\, \frac{\sqrt{PQ}}{(p-q)}\, \mathcal{A} + \frac{PQ}{(p-q)^2}\, \left(u_2^2 + u_3^2\right) + \mathcal{A}^2\, \left(u_1^2 + u_3^2\right) \right] \\ & + \sqrt{2\kappa|u|^2 + u_0}\, \frac{P\, \left(q^4 + \beta^2 - 2\,\beta\, q^2\right) + Q\, \left(p^4 + \beta^2 - 2\,\beta\, p^2\right)}{p^2 - q^2}\,, \end{aligned} \tag{5.11}$$

and the one-form $C$ is

$$\begin{aligned}C = \frac{e^{-\frac{4}{3}\phi}}{\sqrt{2\kappa|u|^2+u_0}}\Bigg\{\Bigg[&\mathcal{A}(u_3\,du_1-u_1\,du_3)+\frac{\sqrt{PQ}(u_3\,du_2-u_2\,du_3)}{(p-q)}\Bigg]+\\ &+u_3\left[\frac{(p-q)\,\mathcal{A}\,u_2-\sqrt{PQ}\,u_1}{(p-q)^2}\right]\left(\sqrt{\frac{Q}{P}}\,dp+\sqrt{\frac{P}{Q}}\,dq\right)+\\ &+dw\left[-\kappa(p-q)\mathcal{A}(u_1^2+u_3^2)+\beta\kappa(p-q)^2(u_1^2+u_3^2)-\kappa\sqrt{PQ}u_1u_2\right]\Bigg\}+\\ +e^{-\frac{4}{3}\phi}\sqrt{2\kappa|u|^2+u_0}\,&\frac{P\,(q^2-\beta)+Q\,(p^2-\beta)}{p^2-q^2}\,dw\,. \end{aligned} \tag{5.12}$$

The reduced metric is then:

$$ds^2_{(10)} = e^{\frac{2}{3}\phi}\left[dx^a\,dx^b\,\eta_{ab}+\frac{1}{\sqrt{2\kappa|u|^2+u_0}}\left(du^i+\epsilon^{ijk}\tilde{A}^j u^k\right)^2+\sqrt{2\kappa|u|^2+u_0}\,d\tilde{s}_3\right] \tag{5.13}$$

with

$$\tilde{A}^1 = 0\,, \qquad \tilde{A}^2 = -\kappa(p-q)dw\,, \qquad \tilde{A}^3 = \frac{1}{(p-q)}\left[\sqrt{\frac{Q}{P}}\,dp+\sqrt{\frac{P}{Q}}\,dq\right], \tag{5.14}$$

and

$$d\tilde{s}_3 = \frac{p^2-q^2}{P}dp^2+\frac{p^2-q^2}{Q}dq^2+\frac{P+Q}{p^2-q^2}\,dw^2\,. \tag{5.15}$$

From these explicit formulae we can now see that $w \to w+c$ is the surviving U(1) isometry of the background, commuting with the $\partial_z$ upon which we reduced the 11-dimensional solution. One can therefore think about the possibility of further reducing the above solution to 9 dimensions along this direction and consider the $T$-dual picture. The interesting fact is that the corresponding Killing vector does not show any fixed point (unless one considers a double root of our $P$ and $Q$ polynomials, which then becomes an essential singularity). Let us conclude that in this reduction we do not produce $NS5$-branes in addition to the $D6$-brane setup discussed in the previous section.

To be explicit, we consider now the above reduction in the case that $\beta$ is chosen such that one can have codimension four singularities at NUT fixed points of the quaternionic manifold, i.e. $\beta$ satisfying (4.12). Doing this we expect to obtain a setup of three intersecting $D6$-branes and we want to analyze the behaviour of our solution when the coordinates approach the fixed point corresponding to one of these branes. The fixed point we will discuss sits at

$$p = r_4\,, \qquad q = r_3\,, \tag{5.16}$$

and we choose to have $r_1 = r_2$, such that the additional codimension six singularities are removed.[1] The first thing to be pointed out is that at such fixed points the string coupling constant vanishes, as the dilaton can be expressed as the square of the Killing isometry, see (5.5), and this latter must go to zero at the fixed points. We can then proceed to the

[1]The analysis for the other fixed point is totally symmetric upon exchange of $p$ and $q$.

analysis of the limit of the reduced 10-dimensional metric and one-form. Before proceeding with the limit, we have to remember that the fixed point is found by fixing the value of two coordinates and that therefore the limiting procedure has to be defined accordingly. Since the surfaces $p = r_4$ or $q = r_3$ already show an irregular behaviour for the $C$ field and the metric, we decided to approach the fixed point in a "diagonal" direction. This means that we took a similar scaling for $p$ and $q$, namely $p = r_4 - x$, $q = r_3 - x$ and then took the limit $x \to 0^+$ for $\kappa > 0$. In this way it can be checked that the dilaton behaviour is linear in $x$

$$e^{\frac{4}{3}\phi} \simeq \frac{\left(3\,r_3^2 + 10\,r_3\,r_4 + 3\,r_4^2\right)^2\,\left(5\,r_3^2 + 6\,r_3\,r_4 + 5\,r_4^2\right)}{128\,\left(r_3 + r_4\right)}\,\sqrt{u_0 + 2\,\kappa\,|u|^2}\,\kappa\,x + O(x^2)\,, \quad (5.17)$$

and all the above quantities are positive. The same limit in the metric shows the expected behaviour for a $D6$-brane geometry, taking care of the fact that in our parametrization the internal and transverse space are not expressed through cartesian coordinates. As a matter of fact, it can be shown that the leading behaviour is given by

$$g_{pp} \sim g_{qq} \sim \frac{1}{\sqrt{x}}\,, \quad (5.18)$$

whereas

$$g_{ww} \sim g_{w.v.} \sim \sqrt{x}\,. \quad (5.19)$$

Here we called $g_{w.v.}$ the world-volume metric and $p$, $q$ and $w$ are the transverse coordinates. The behaviour of the metric in the $w$ direction is different from the standard one shown by $p$ and $q$ because $w$ is an angular coordinate parametrizing the U(1) isometry of the resulting metric and therefore one has to add further scaling coming from the radial direction. Again, as expected, the two-form field strength $F$ shows a diverging behaviour in the $p$ and $q$ directions

$$F_{p\mu} \sim F_{q\mu} \sim \frac{1}{x}\,, \quad (5.20)$$

whereas all the other components go to some constant value. In line of principle one could now also derive the exact number of $D6$-branes sitting at such fixed point by integrating the $F$-form along the collapsing two-cycle of the metric. Unfortunately, as already shown by the dilaton expression (5.17), the definitions of $F$ are highly complex in our coordinate system and therefore we decided not to perform such computation.

## 6. Conclusion

In the paper, we discussed in detail the metric of a new 7-manifold with $G_2$ holonomy. This space is topologically a $\mathbb{R}^3$ bundle over a quaternionic space with a $U(1) \times U(1)$ isometry group and is determined by a single fourth order polynomial. A generic Killing vector has fixed points of various co-dimension, but most interesting are co-dimension 4 fixed points that give D6-branes upon dimensional reduction. As we discussed in detail, this requires to pick specific Killing vectors and we found exactly two possibilities to obtain D6-branes. In order to avoid additional co-dimension 6 fixed points one has to equalize two roots of the fourth order polynomial. The co-dimension four fixed point set consist of two components and we concluded therefore that there are three stacks of D6-branes, where two of the stacks have equal number of branes.

Following the arguments given in the mathematical literature [43], it is very tempting to relate this space to the weighted projective space. In fact, the four roots of the fourth order polynomial sum up to zero and hence are parameterized by three (quantized) parameters, which should be related to the three weights of $\mathbb{WCP}^2_{abc}$. In order to avoid co-dimension 6 fixed points we had to identify two roots and the remaining two parameters where quantized. As a consequence, the number of 6-branes in two stacks agree and we expect a gauge group $\mathrm{SU}(m)\times\mathrm{SU}(m)\times\mathrm{SU}(n)$, where in the deformed case the higgsing should be done in such a way that the product of two equal gauge groups survives; because the two components of the fixed point set are related to the same number of 6-branes. At the moment, these conclusions are more speculative and further investigations are necessary.

## Acknowledgments

We would like to thank Andreas Brandhuber and Sergei Gukov for valuable discussions. The work of K.B. is supported by a Heisenberg grant of the DFG and G. D. acknowledges the financial support provided through the European Community's Human Potential Programme under contract HPRN-CT-2000-00131 quantum spacetime. S.M. would like to thank Alexander von Humboldt Foundation for financial support during the initial stage of this work. S.M. also acknowledges the facilities provided by the computer centre, I.O.P. Bhubaneswar.¡

## References

[1] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Kaluza-Klein supergravity*, *Phys. Rept.* **130** (1986) 1.

[2] G. Papadopoulos and P.K. Townsend, *Compactification of $D = 11$ supergravity on spaces of exceptional holonomy*, *Phys. Lett.* **B 357** (1995) 300 [hep-th/9506150].

[3] B.S. Acharya, *On realising $N = 1$ super Yang-Mills in M-theory*, hep-th/0011089.

[4] M. Atiyah, J.M. Maldacena and C. Vafa, *An M-theory flop as a large-$N$ duality*, *J. Math. Phys.* **42** (2001) 3209 [hep-th/0011256].

[5] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *M3-branes, $G_2$ manifolds and pseudo-supersymmetry*, *Nucl. Phys.* **B 620** (2002) 3 [hep-th/0106026].

[6] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, *Gauge theory at large-$N$ and new $G_2$ holonomy metrics*, *Nucl. Phys.* **B 611** (2001) 179 [hep-th/0106034].

[7] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of $G_2$ holonomy*, hep-th/0107177.

[8] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *Cohomogeneity one manifolds of* Spin(7) *and $G_2$ holonomy*, *Phys. Rev.* **D 65** (2002) 106004 [hep-th/0108245].

[9] S. Gukov and J. Sparks, *M-theory on* Spin(7) *manifolds 1*, *Nucl. Phys.* **B 625** (2002) 3 [hep-th/0109025].

[10] E. Witten, *Anomaly cancellation on $G_2$ manifolds*, hep-th/0108165.

[11] B. Acharya and E. Witten, *Chiral fermions from manifolds of $G_2$ holonomy*, hep-th/0109152.

[12] G. Curio, B. Körs and D. Lüst, *Fluxes and branes in type-II vacua and M-theory geometry with $G_2$ and* Spin(7) *holonomy*, *Nucl. Phys.* **B 636** (2002) 197 [hep-th/0111165].

[13] A. Brandhuber, *$G_2$ holonomy spaces from invariant three-forms*, *Nucl. Phys.* **B 629** (2002) 393 [hep-th/0112113].

[14] S. Gukov and D. Tong, *D-brane probes of special holonomy manifolds and dynamics of $N = 1$ three-dimensional gauge theories*, *J. High Energy Phys.* **04** (2002) 050 [hep-th/0202126].

[15] K. Behrndt, *Singular 7-manifolds with $G_2$ holonomy and intersecting 6-branes*, *Nucl. Phys.* **B 635** (2002) 158 [hep-th/0204061].

[16] C.I. Lazaroiu and L. Anguelova, *M-theory compactifications on certain 'toric' cones of $G_2$ holonomy*, hep-th/0204249.

[17] L. Anguelova and C.I. Lazaroiu, *M-theory on 'toric' $G_2$ cones and its type-II reduction*, hep-th/0205070.

[18] P. Berglund and A. Brandhuber, *Matter from $G_2$ manifolds*, hep-th/0205184.

[19] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *Bianchi IX self-dual Einstein metrics and singular $G_2$ manifolds*, hep-th/0206151.

[20] R. Blumenhagen, L. Görlich, B. Körs and D. Lüst, *Noncommutative compactifications of type-I strings on tori with magnetic background flux*, *J. High Energy Phys.* **10** (2000) 006 [hep-th/0007024].

[21] G. Aldazabal, S. Franco, L.E. Ibáñez, R. Rabadán and A.M. Uranga, *Intersecting brane worlds*, *J. High Energy Phys.* **02** (2001) 047 [hep-ph/0011132].

[22] G. Aldazabal, S. Franco, L.E. Ibáñez, R. Rabadán and A.M. Uranga, *$D = 4$ chiral string compactifications from intersecting branes*, *J. Math. Phys.* **42** (2001) 3103 [hep-th/0011073].

[23] R. Blumenhagen, B. Körs and D. Lüst, *Type I strings with F- and B-flux*, *J. High Energy Phys.* **02** (2001) 030 [hep-th/0012156].

[24] L.E. Ibáñez, F. Marchesano and R. Rabadán, *Getting just the standard model at intersecting branes*, *J. High Energy Phys.* **11** (2001) 002 [hep-th/0105155].

[25] R. Blumenhagen, B. Körs, D. Lüst and T. Ott, *The standard model from stable intersecting brane world orbifolds*, *Nucl. Phys.* **B 616** (2001) 3 [hep-th/0107138].

[26] M. Cvetič, G. Shiu and A.M. Uranga, *Three-family supersymmetric standard like models from intersecting brane worlds*, *Phys. Rev. Lett.* **87** (2001) 201801 [hep-th/0107143].

[27] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral type-II orientifold constructions as M-theory on $G_2$ holonomy spaces*, hep-th/0111179.

[28] R. Blumenhagen, V. Braun, B. Körs and D. Lüst, *Orientifolds of K3 and Calabi-Yau manifolds with intersecting D-branes*, *J. High Energy Phys.* **07** (2002) 026 [hep-th/0206038].

[29] R.L. Bryant and S.M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, *Duke Math. J.* **58** (1989) 829.

[30] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on $S^3$, $R^3$ and $R^4$ bundles*, *Commun. Math. Phys.* **127** (1990) 529.

[31] J. Wolf, *Complex homogenous contact manifolds and quaternionic symmetric spaces*, *J. Math. Mech.* **14** (1965) 1033.

[32] B. de Wit, B. Kleijn and S. Vandoren, *Superconformal hypermultiplets*, *Nucl. Phys.* **B 568** (2000) 475 [hep-th/9909228].

[33] B. de Wit, M. Roček and S. Vandoren, *Hypermultiplets, hyperkaehler cones and quaternion-kaehler geometry*, *J. High Energy Phys.* **02** (2001) 039 [hep-th/0101161].

[34] B. de Wit, M. Roček and S. Vandoren, *Gauging isometries on hyperkaehler cones and quaternion-Kaehler manifolds*, *Phys. Lett.* **B 511** (2001) 302 [hep-th/0104215].

[35] J.F. Plebanski, *A class of solutions of Einstein-Maxwell equations*, *Ann. Phys. (NY)* **90** (1975) 196.

[36] J.F. Plebanski and M. Demianski, *Rotating, charged and uniformly accelerating mass in general relativity*, *Ann. Phys. (NY)* **98** (1976) 98.

[37] B. de Wit and H. Nicolai, *The parallelizing $S_7$ torsion in gauged $N = 8$ supergravity*, *Nucl. Phys.* **B 231** (1984) 506.

[38] A. Bilal, J.-P. Derendinger and K. Sfetsos, *(Weak) $G_2$ holonomy from self-duality, flux and supersymmetry*, *Nucl. Phys.* **B 628** (2002) 112 [hep-th/0111274].

[39] S. Chiossi and S. Salamon, *The intrinsic torsion of* SU(3) *and $G_2$ structures*, math.DG/0202282.

[40] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *New complete non-compact* Spin(7) *manifolds*, *Nucl. Phys.* **B 620** (2002) 29 [hep-th/0103155].

[41] R. Bryant, *Bochner Kaehler metrics*, math.DG/0003099.

[42] V. Apostolov and P. Gauduchon, *Selfdual Einstein hermitean four manifolds*, math.DG/0003162.

[43] D.M.J. Calderbank and H. Pedersen, *Selfdual Einstein metrics with torus symmetry*, math.DG/0105263.

[44] P.-Y. Casteill, E. Ivanov and G. Valent, $\mathrm{U}(1) \times \mathrm{U}(1)$ *quaternionic metrics from harmonic superspace*, *Nucl. Phys.* **B 627** (2002) 403 [hep-th/0110280].

[45] P.-Y. Casteill, E. Ivanov and G. Valent, *Quaternionic extension of the double Taub-NUT metric*, *Phys. Lett.* **B 508** (2001) 354 [hep-th/0104078].

[46] N. Alonso-Alberca, P. Meessen and T. Ortin, *Supersymmetry of topological Kerr-Newman-Taub-NUT-adS spacetimes*, *Class. and Quant. Grav.* **17** (2000) 2783 [hep-th/0003071].

[47] A. Sen, *Strong coupling dynamics of branes from M-theory*, *J. High Energy Phys.* **10** (1997) 002 [hep-th/9708002].

[48] D. Kramer, E. Herlt, M. MacCallum, and H. Stephani, *Exact solutions of Einstein's field equations*, Cambridge University Press 1979.

[49] K. Behrndt and G. Dall'Agata, *Vacua of $N = 2$ gauged supergravity derived from non-homogenous quaternionic spaces*, *Nucl. Phys.* **B 627** (2002) 357 [hep-th/0112136].

[50] A. Ceresole, G. Dall'Agata, R. Kallosh and A. Van Proeyen, *Hypermultiplets, domain walls and supersymmetric attractors*, *Phys. Rev.* **D 64** (2001) 104006 [hep-th/0104056].

[51] D.V. Alekseevsky, V. Cortes, C. Devchand and A. Van Proeyen, *Flows on quaternionic-Kaehler and very special real manifolds*, hep-th/0109094.

[52] R.M. Wald, *General relativity*, Chicago University Press, Chicago 1984.

[53] S.W. Hawking and C.J. Hunter, *Gravitational entropy and global structure*, *Phys. Rev.* **D 59** (1999) 044025 [hep-th/9808085].

[54] G.W. Gibbons and S.W. Hawking, *Classification of gravitational instanton symmetries*, *Commun. Math. Phys.* **66** (1979) 291.

JHEP08(2002)027